Pergamon

Int. J. Solids Structures Vol. 33, No. 26, pp. 3865–3880, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00218–9

INTERACTION BETWEEN A CRACK AND A CIRCULAR ELASTIC INCLUSION UNDER REMOTE UNIFORM HEAT FLOW

C. K. CHAO and J. Y. LEE
Department of Mechanical Engineering, National Taiwan Institute of Technology, Taipei, Taiwan, 106, Republic of China

(*Received* 20 *March* 1995; *in revised form* 13 *September* 1995)

Abstract—The plane interaction problem for a circular elastic inclusion embedded in an elastic matrix with an arbitrarily oriented crack, located either in the matrix or in the inclusion under remote uniform heat flow, is considered. By using the complex variable theory and the existing solutions for dislocation functions, the thermoelastic problem of a crack in the form of an arbitrary shape in the vicinity of the interface is formulated. The integral equations for a line crack are then obtained as a system of singular integral equations with logarithmic singular kernels. The stress intensity factors, which can properly reflect the interaction between a crack and a circular inclusion, are obtained in terms of the values of the density functions of the integral equations. Several numerical examples are given to demonstrate the effects of geometrical parameters and material property combinations on the strength of the thermal stress singularity. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The problem of plane thermoelasticity for two bonded half-plane media with cracks of various shapes has been recently solved by Chao and Shen (1995). It was shown that the solution of the problem with cracks embedded in one of each medium is related to the corresponding homogeneous problem. This is merely done by the method of analytical continuation which allows us to obtain the solution associated with bimaterial media (or a half-plane medium) from the result associated with the homogeneous problem by a simple algebraic substitution. The purpose of the present paper is to extend the above mentioned methodology to the case where a circular thermoelastic inclusion is perfectly bonded to the matrix with cracks located either in the matrix or in the inclusion.

Problems of the crack interaction with an elastic inclusion have been extensively investigated by many researchers. Early work by Dundurs and Mura (1964) derived the analytical solution of the problem of a circular elastic inclusion near an edge dislocation in terms of Airy's stress potentials. Since then a number of crack problems interacted with elastic inclusions have received considerable interest such as Dundurs and Sendeckyj (1965); Atkinson (1972); Erdogan and Gupta (1975); Hutchinson (1987); Patton and Santare (1993); Li and Chudnovsky (1994). All the aforementioned studies have concentrated on crack problems interacting with elastic inclusions under isothermal conditions. In this paper, the problem of cracks interacting with an elastic inclusion under remote uniform heat flow is solved. The proposed method is based upon the concept of continuation theorem, similar to the Kelvin transformation applied by Honein and Herrmann (1988, 1990), which enables us to establish the interindependent relations of the related complex potentials. Moreover, the singular equations with a logarithmic singular kernel (Chen, 1990; Chao and Shen, 1995) are derived and solved numerically in a straightforward manner. Some examples of a circular inclusion interacting with an arbitrarily oriented crack under remote uniform heat flow are given to illustrate the use of the present approach. The results presented here may assist in studies of cracks interaction with an array of microdefects and/or inclusions under thermal loadings.

2. TEMPERATURE FIELD

Consider a circular inclusion of the radius a perfectly bonded to an infinite matrix with cracks located either in the matrix or in the inclusion. As shown in Fig. 1, the regions occupied by the matrix ($|z| > a$) and the inclusion ($|z| < a$) will be referred to as regions S_1 and S_2 respectively, and the quantities associated with these regions will be denoted by the corresponding subscripts. For two-dimensional steady state heat conduction problems, the temperature function associated with the matrix and the inclusion can be expressed in terms of analytical functions $g_1'(z)$ and $g_2'(z)$, respectively. In the present study, the resultant heat flux Q_j and temperature T_j for each medium, both used to formulate the boundary condition along the interface, are expressed in terms of complex potentials as

$$Q_j = \int (q_{xj}\,\mathrm{d}y - q_{yj}\,x) = -k_j \,\mathrm{Im}\,[g_j'(z)] \tag{1}$$

$$T_j = \mathrm{Re}\,[g_j'(z)] \tag{2}$$

where Re and Im stand for the real part and imaginary part of the bracketed expression, respectively. The quantities q_{xj} and q_{yj} in eqn (1) represent the components of heat flux in the x- and y-directions, respectively, and k_j denotes the heat conductivities.

2.1. *Cracks in the matrix*

If there exist cracks in the matrix, the temperature functions in the matrix and in the inclusion can be respectively expressed as

$$g_1'(z) = g_0'(z) + \underset{\sim}{g}_1'(z) \tag{3}$$

$$g_2'(z) = \underset{\sim}{g}_2'(z) \tag{4}$$

where $g_0'(z)$ stands for the temperature function associated with the unperturbed field which is related to the solution of cracks in the homogeneous media. $\underset{\sim}{g}_1'(z)$ (or $\underset{\sim}{g}_2'(z)$) is the temperature function associated with the perturbed field of matrix (or inclusion) which is holomorphic in region S_1 (or S_2). Since the matrix and the inclusion are assumed to be perfectly bonded along the interface, both the temperature and resultant heat flow should be continuous across the interface $z = \sigma = ae^{i\theta}$, resulting in

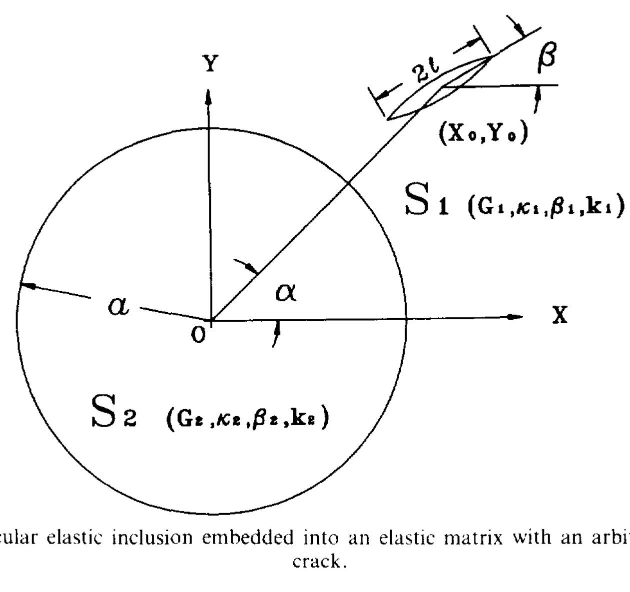


Fig. 1. A circular elastic inclusion embedded into an elastic matrix with an arbitrarily oriented crack.

$$[g'_0(\sigma)+\underset{\sim}{g}'_1(\sigma)]+[\overline{g'_0(\sigma)}+\overline{\underset{\sim}{g}'_1(\sigma)}] = [g'_2(\sigma)+\overline{\underset{\sim}{g}'_2(\sigma)}] \tag{5}$$

$$ik_1\{[g'_0(\sigma)+\underset{\sim}{g}'_1(\sigma)]-[\overline{g'_0(\sigma)}+\overline{\underset{\sim}{g}'_1(\sigma)}]\} = ik_2[g'_2(\sigma)-\overline{\underset{\sim}{g}'_2(\sigma)}] \tag{6}$$

where an overbar denotes the complex conjugate. By applying the continuation theorem, it is convenient to introduce a new set of complex potentials $\theta_j(z)$ $(j = 1,2)$ which is holomorphic in the entire domain including the interface as

$$\theta_1(z) = \underset{\sim}{g}'_1(z)-\bar{\underset{\sim}{g}}'_2(a^2/z)+\bar{g}'_0(a^2/z) \tag{7}$$

$$\theta_2(z) = ik_1\underset{\sim}{g}'_1(z)+ik_2\bar{\underset{\sim}{g}}'_2(a^2/z)-ik_1\bar{g}'_0(a^2/z) \tag{8}$$

for $z \in S_1$, and

$$\theta_1(z) = \underset{\sim}{g}'_2(z)-\bar{\underset{\sim}{g}}'_1(a^2/z)-g'_0(z) \tag{9}$$

$$\theta_2(z) = ik_2\underset{\sim}{g}'_2(z)+ik_1\bar{\underset{\sim}{g}}'_1(a^2/z)-ik_1g'_0(z) \tag{10}$$

for $z \in S_2$. Since $\theta_j(z)$ are now holomorphic and single-valued in the whole plane including the point at infinity, according to Liouville's theorem, $\theta_j(z)$ is considered as a constant. However, the constant functions $\theta_j(z)$ can be treated as a reference temperature and which can thus be assumed to be zero without loss in generality. With this result, the final expression of the temperature functions becomes

$$g'_1(z) = g'_0(z)+\frac{k_1-k_2}{k_1+k_2}\bar{g}'_0(a^2/z) \tag{11}$$

$$g'_2(z) = \frac{2k_1}{k_1+k_2}g'_0(z) \tag{12}$$

2.2. *Cracks in the inclusion*

When cracks are located inside the inclusion, the temperature functions can be written as

$$g'_1(z) = \underset{\sim}{g}'_1(z) \tag{13}$$

$$g'_2(z) = g'_0(z)+\underset{\sim}{g}'_2(z) \tag{14}$$

By applying the interface continuity conditions and the method of analytical continuation, similar to the derivations given in eqns (5)–(10), the final expression of the temperature functions becomes

$$g'_1(z) = \frac{2k_2}{k_1+k_2}g'_0(z) \tag{15}$$

$$g'_2(z) = g'_0(z)+\frac{k_2-k_1}{k_1+k_2}\bar{g}'_0(a^2/z) \tag{16}$$

3. THERMAL STRESS FIELD

For the two-dimensional theory of thermoelasticity, the components of the displacement and traction force can be expressed in terms of two stress functions $\phi_j(z)$, $\psi_j(z)$ and a temperature function $g_j(z)$ as (Bogdanoff, 1954)

$$2G_j(u_j+iv_j) = \kappa_j\phi_j(z)-z\overline{\phi_j'(z)}-\overline{\psi_j(z)}+2G_j\beta_jg_j(z) \tag{17}$$

$$-Y_j+iX_j = \phi_j(z)+z\overline{\phi_j'(z)}+\overline{\psi_j(z)} \tag{18}$$

where G_j is the shear modulus, and $\kappa_j = 3-v_j/1+v_j$, $\beta_j = \alpha_j$ for plane stress and $\kappa_j = 3-4v_j$, $\beta_j = (1+v_j)\alpha_j$ for plane strain with v_j being the Poisson's ratio and α_j the thermal expansion coefficients. The components of stress in polar coordinates are related to $\phi_j(z)$ and $\psi_j(z)$ by

$$[\sigma_{rr}+\sigma_{\theta\theta}]_j = 4\mathrm{Re}\,[\phi_j'(z)] \tag{19}$$

$$[\sigma_{\theta\theta}-\sigma_{rr}+2i\sigma_{r\theta}]_j = 2[\bar{z}\phi_j''(z)+\psi_j'(z)]e^{2i\theta} \tag{20}$$

3.1. *Cracks in the matrix*

We first consider a circular inclusion perfectly bonded to an infinite matrix in which the cracks are located. The stress functions in the matrix and in the inclusion can be written respectively as

$$\phi_1(z) = \phi_0(z)+\underset{\sim}{\phi_1}(z),\quad \psi_1(z) = \psi_0(z)+\underset{\sim}{\psi_1}(z) \tag{21}$$

and

$$\phi_2(z) = \underset{\sim}{\phi_2}(z),\quad \psi_2(z) = \underset{\sim}{\psi_2}(z) \tag{22}$$

where $\phi_0(z)$ and $\psi_0(z)$ represent the stress functions associated with the unperturbed field which are related to the solutions of cracks in the homogeneous media while $\underset{\sim}{\phi_1}(z)$, $\underset{\sim}{\psi_1}(z)$ (or $\underset{\sim}{\phi_2}(z)$, $\underset{\sim}{\psi_2}(z)$) are the functions corresponding to the perturbed field of matrix (or inclusion). Since the matrix and the inclusion are assumed to be perfectly bonded along the common boundary, both the displacement and traction force must be continuous across the interface $z = \sigma = ae^{i\theta}$. This can then lead to

$$\phi_0(\sigma)+\underset{\sim}{\phi_1}(\sigma)+\sigma\overline{\phi_0'(\sigma)}+\sigma\overline{\underset{\sim}{\phi_1'}(\sigma)}+\overline{\psi_0(\sigma)}+\overline{\underset{\sim}{\psi_1}(\sigma)} = \underset{\sim}{\phi_2}(\sigma)+\sigma\overline{\underset{\sim}{\phi_2'}(\sigma)}+\overline{\underset{\sim}{\psi_2}(\sigma)} \tag{23}$$

and

$$\frac{1}{2G_1}\{\kappa_1[\phi_0(\sigma)+\underset{\sim}{\phi_1}(\sigma)]-\sigma\overline{\phi_0'(\sigma)}-\sigma\overline{\underset{\sim}{\phi_1'}(\sigma)}-\overline{\psi_0(\sigma)}-\overline{\underset{\sim}{\psi_1}(\sigma)}\}+\beta_1g_0(\sigma)$$

$$+\frac{(k_1-k_2)\beta_1}{k_1+k_2}\bar{f}_0(a^2/\sigma) = \frac{1}{2G_2}\{\kappa_2\underset{\sim}{\phi_2}(\sigma)-\sigma\overline{\underset{\sim}{\phi_2'}(\sigma)}-\overline{\underset{\sim}{\psi_2}(\sigma)}\}+\frac{2k_1\beta_2}{k_1+k_2}g_0(\sigma) \tag{24}$$

where

$$f_0(z) = -a^2\int g_0'(z)\frac{\mathrm{d}z}{z^2}$$

Using the properties of holomorphic functions and applying the method of analytical continuation, a new set of complex potentials $\omega_j(z)$, which are holomorphic in the entire domain including the interface boundary, is introduced as

$$\omega_1(z) = \underset{\sim}{\phi}_1(z) + z\bar{\phi}_0'(a^2/z) - z\overline{\phi_0'(0)} + \bar{\psi}_0(a^2/z) - z\underset{\sim}{\bar{\phi}}_2'(a^2/z) + z\overline{\underset{\sim}{\phi}_2'(0)} - \underset{\sim}{\bar{\psi}}_2(a^2/z) \quad (25)$$

$$\omega_2(z) = \frac{\kappa_1}{2G_1}\underset{\sim}{\phi}_1(z) - \frac{1}{2G_1}[z\bar{\phi}_0'(a^2/z) - z\overline{\phi_0'(0)} + \bar{\psi}_0(a^2/z)] + \frac{1}{2G_2}[z\underset{\sim}{\bar{\phi}}_2'(a^2/z) - z\overline{\underset{\sim}{\phi}_2'(0)} + \underset{\sim}{\bar{\psi}}_2(a^2/z)] + \frac{(k_1-k_2)\beta_1}{k_1+k_2}\bar{f}_0(a^2/z) \quad (26)$$

for $z \in S_1$, and

$$\omega_1(z) = \underset{\sim}{\phi}_2(z) - \phi_0(z) - z\underset{\sim}{\bar{\phi}}_1'(a^2/z) - z\overline{\phi_0'(0)} - \underset{\sim}{\bar{\psi}}_1(a^2/z) + z\overline{\underset{\sim}{\phi}_2'(0)} \quad (27)$$

$$\omega_2(z) = \frac{\kappa_2}{2G_2}\underset{\sim}{\phi}_2(z) - \frac{1}{2G_2}z\overline{\underset{\sim}{\phi}_2'(0)} + \frac{1}{2G_1}[z\underset{\sim}{\bar{\phi}}_1'(a^2/z) + z\overline{\phi_0'(0)} + \underset{\sim}{\bar{\psi}}_1(a^2/z) - \kappa_1\phi_0(z)] + \left(\frac{2k_1}{k_1+k_2}\beta_2 - \beta_1\right)g_0(z) \quad (28)$$

for $z \in S_2$. By Liouville's theorem, we have $\omega_j(z) \equiv$ constant. However, the constant functions $\omega_j(z)$ correspond to a rigid-body motion which may be neglected, we have $\omega_j(z) \equiv 0$. Based on this result, the final expression of the stress functions becomes

$$\phi_1(z) = \phi_0(z) + \gamma_3[z\bar{\phi}_0'(a^2/z) - z\overline{\phi_0'(0)} + \bar{\psi}_0(a^2/z)] + \gamma_2\bar{f}_0(a^2/z) \quad (29)$$

$$\psi_1(z) = \psi_0(z) + \gamma_1\bar{\phi}_0(a^2/z) + \gamma_3\frac{a^4}{z^3}\left[\bar{\psi}_0'(a^2/z) + z\bar{\phi}_0''(a^2/z) - \frac{z^2}{a^2}\bar{\phi}_0'(a^2/z)\right] + \gamma_4\bar{g}_0(a^2/z) + \frac{a^4}{z^3}\gamma_2\bar{f}_0'(a^2/z) + \frac{a^2}{z}\left\{\left[\frac{1+\gamma_1}{1-\gamma_3^*}\gamma_3^* - \gamma_3\right]\overline{\phi_0'(0)} + \left[\frac{1+\gamma_1}{1-\gamma_3^*} - 1\right]\phi_0'(0) + \frac{\gamma_4}{1-\gamma_3^*}[\gamma_3^*\overline{g_0'(0)} + g_0'(0)]\right\} \quad (30)$$

$$\phi_2(z) = (1+\gamma_1)\phi_0(z) + \gamma_4 g_0(z) + \frac{\gamma_3^*}{1-\gamma_3^{*2}}z\{(1+\gamma_1)[\gamma_3^*\phi_0'(0) + \overline{\phi_0'(0)}] + \gamma_4[\gamma_3^* g_0'(0) + \overline{g_0'(0)}]\} \quad (31)$$

$$\psi_2(z) = (1+\gamma_3)\left[\frac{a^2}{z}\phi_0'(z) - \frac{a^2}{z}\phi_0'(0) + \psi_0(z)\right] - \frac{a^2}{z}\phi_2'(z) + \gamma_2 f_0(z) + \frac{a^2}{z}\phi_2'(0) \quad (32)$$

where

$$\gamma_1 = \frac{\kappa_1 G_2 - \kappa_2 G_1}{\kappa_2 G_1 + G_2}, \quad \gamma_2 = \frac{2G_1G_2\beta_1}{\kappa_1 G_2 + G_1}\frac{k_2-k_1}{k_1+k_2}, \quad \gamma_3 = \frac{G_2 - G_1}{\kappa_1 G_2 + G_1},$$
$$\gamma_4 = \frac{2G_1G_2}{\kappa_2 G_1 + G_2}\left(\beta_1 - \frac{2k_1}{k_1+k_2}\beta_2\right), \quad \gamma_3^* = \frac{G_1 - G_2}{\kappa_2 G_1 + G_2} \quad (33)$$

For the special case of isothermal elasticity, the expressions provided in eqns (29)–(32) will be simplified to the results given by Honein and Herrmann (1990) if one assumes $\gamma_2 = \gamma_4 = 0$ and $\phi'_0(0) = \overline{\phi'_0(0)}$ in the above equations.

3.2. *Cracks in the inclusion*

If all cracks are located inside the inclusion, the stress functions in the matrix and in the inclusion can now be written respectively as

$$\phi_1(z) = \underset{\sim}{\phi}_1(z), \quad \psi_1(z) = \underset{\sim}{\psi}_1(z) \tag{34}$$

and

$$\phi_2(z) = \phi_0(z)+\underset{\sim}{\phi}_2(z), \quad \psi_2(z) = \psi_0(z)+\underset{\sim}{\psi}_2(z) \tag{35}$$

Based upon the expressions given in eqns (34) and (35), the interface continuity conditions lead to

$$\underset{\sim}{\phi}_1(\sigma)+\sigma\overline{\underset{\sim}{\phi}'_1(\sigma)}+\overline{\underset{\sim}{\psi}_1(\sigma)} = \phi_0(\sigma)+\underset{\sim}{\phi}_2(\sigma)+\sigma[\overline{\phi'_0(\sigma)}+\overline{\underset{\sim}{\phi}'_2(\sigma)}]+\overline{\psi_0(\sigma)}+\overline{\underset{\sim}{\psi}_2(\sigma)} \tag{36}$$

and

$$\frac{1}{2G_1}[\kappa_1\underset{\sim}{\phi}_1(\sigma)-\sigma\overline{\underset{\sim}{\phi}'_1(\sigma)}-\overline{\underset{\sim}{\psi}_1(\sigma)}]+\frac{2k_2\beta_1}{k_1+k_2}g_0(\sigma)$$
$$= \frac{1}{2G_2}\{\kappa_2[\phi_0(\sigma)+\underset{\sim}{\phi}_2(\sigma)]-\sigma[\overline{\phi'_0(\sigma)}+\overline{\underset{\sim}{\phi}'_2(\sigma)}]-\overline{\psi_0(\sigma)}-\overline{\underset{\sim}{\psi}_2(\sigma)}\}$$
$$+\beta_2 g_0(\sigma)+\frac{(k_2-k_1)\beta_2}{k_1+k_2}\bar{f}_0(a^2/\sigma) \tag{37}$$

Applying the method of analytical continuation, a new set of complex potentials $\omega_j(z)$ is introduced as

$$\omega_1(z) = \underset{\sim}{\phi}_1(z)-z[\bar{\underset{\sim}{\phi}}'_2(a^2/z)-\overline{\underset{\sim}{\phi}'_2(0)}]-\phi_0(z)-\bar{\underset{\sim}{\psi}}_2(a^2/z) \tag{38}$$

$$\omega_2(z) = \frac{\kappa_1}{2G_1}\underset{\sim}{\phi}_1(z)+\frac{1}{2G_2}[z\bar{\underset{\sim}{\phi}}'_2(a^2/z)-z\overline{\underset{\sim}{\phi}'_2(0)}-\kappa_2\phi_0(z)+\bar{\underset{\sim}{\psi}}_2(a^2/z)]$$
$$+\left(\frac{2k_2}{k_1+k_2}\beta_1-\beta_2\right)g_0(z) \tag{39}$$

for $z \in S_1$, and

$$\omega_1(z) = \underset{\sim}{\phi}_2(z)-z\bar{\underset{\sim}{\phi}}'_1(a^2/z)-\bar{\underset{\sim}{\psi}}_1(a^2/z)+z\overline{\underset{\sim}{\phi}'_2(0)}+z\bar{\phi}'_0(a^2/z)+\bar{\psi}_0(a^2/z) \tag{40}$$

$$\omega_2(z) = \frac{\kappa_2}{2G_2}\underset{\sim}{\phi}_2(z)+\frac{1}{2G_1}[z\bar{\underset{\sim}{\phi}}'_1(a^2/z)+\bar{\underset{\sim}{\psi}}_1(a^2/z)]-\frac{1}{2G_2}[z\overline{\underset{\sim}{\phi}'_2(0)}+z\bar{\phi}'_0(a^2/z)$$
$$+\bar{\psi}_0(a^2/z)]+\frac{(k_2-k_1)\beta_2}{k_1+k_2}\bar{f}_0(a^2/z) \tag{41}$$

for $z \in S_2$. Based on the similar reasons explained in the previous approach, we have $\omega_j(z) \equiv 0$. With this result, the stress functions finally become

$$\phi_1(z) = (1+\gamma_1^*)\phi_0(z)+\gamma_4^* g_0(z) \tag{42}$$

$$\psi_1(z) = (1+\gamma_1^*)\psi_0(z)+(\gamma_3^*-\gamma_1^*)\left[\psi_0(z)+\frac{a^2}{z}\phi_0'(z)\right]+\frac{a^2}{z}\frac{1}{1-\gamma_3^*}[\gamma_3^{*2}\overline{(\hat{\psi}_0)'(0)}$$
$$+\gamma_3^*(\hat{\psi}_0)'(0)+\gamma_2^*\gamma_3^*\overline{(\hat{f}_0)'(0)}+\gamma_2^*(\hat{f}_0)'(0)]+\gamma_2^* g_0(z)-\frac{a^2}{z}\gamma_4^* g_0'(z) \tag{43}$$

$$\phi_2(z) = \phi_0(z)+\gamma_3^*[\bar{\psi}_0(a^2/z)+z\bar{\phi}_0'(a^2/z)]+\frac{\gamma_3^* z}{1-\gamma_3^{*2}}[\gamma_3^{*2}(\hat{\psi}_0)'(0)$$
$$+\gamma_3^*\overline{(\hat{\psi}_0)'(0)}+\gamma_2^*\gamma_3^*(\hat{f}_0)'(0)+\gamma_2^*\overline{(\hat{f}_0)'(0)}]+\gamma_2^*\bar{f}_0(a^2/z) \tag{44}$$

$$\psi_2(z) = \psi_0(z)+\gamma_1^*\bar{\phi}_0(a^2/z)+\gamma_3^*\frac{a^4}{z^3}\left[\bar{\psi}_0'(a^2/z)-\frac{z^2}{a^2}\bar{\phi}_0'(a^2/z)+z\bar{\phi}_0''(a^2/z)\right]$$
$$+\frac{a^2}{z}[\gamma_3^*(\hat{\psi}_0)'(0)+\gamma_2^*(\hat{f}_0)'(0)]+\gamma_4^*\bar{g}_0(a^2/z)+\gamma_2^*\frac{a^4}{z^3}\bar{f}_0'(a^2/z) \tag{45}$$

where $\hat{\psi}_0(z) = \bar{\psi}_0(a^2/z)$, $\hat{f}_0(z) = \bar{f}_0(a^2/z)$ and the constants γ_1^*, γ_2^*, γ_3^*, γ_4^* are obtained from γ_1, γ_2, γ_3, γ_4, respectively, by interchanging the material properties of matrix and inclusion. If all terms concerning temperature effects such as $g_0(z)$ and $\bar{f}_0(z)$ are removed, eqns (42)–(45) are found to be identical to those for isothermal elasticity problem obtained by Honein and Herrmann (1990) except the terms of logarithmic singularities. Note that the existence of logarithmic singularities in the paper (Honein and Herrmann, 1990) results from the discontinuity in the displacement or in the resultant force along the circular boundary. However, the terms of logarithmic singularities would not appear in the crack problem since the requirement of single-valued displacements when enclosing the crack surface should be satisfied.

4. INTEGRAL REPRESENTATION FOR A SINGLE INSULATED CRACK

We first consider a single insulated crack L to be situated in the matrix under remote uniform heat flow. The corresponding homogeneous solutions associated with a single crack can be expressed in terms of distributed temperature dislocations and edge dislocations along the crack surface as (Sekine, 1977)

$$g_0'(z) = -\frac{i}{2\pi}\int_L b_0(s)\log(z-t)\,\mathrm{d}s \tag{46}$$

$$\phi_0(z) = \frac{iG_1}{\pi(1+\kappa_1)}\int_L [b_1(s)+ib_2(s)]\log(z-t)\,\mathrm{d}s \tag{47}$$

$$\psi_0(z) = \frac{-iG_1}{\pi(1+\kappa_1)}\int_L [b_1(s)-ib_2(s)]\log(z-t)\,\mathrm{d}s-\frac{iG_1}{\pi(1+\kappa_1)}\int_L \frac{[b_1(s)+ib_2(s)]\bar{t}}{z-t}\mathrm{d}s \tag{48}$$

where $b_0(s)$ indicates the strength of the temperature dislocation and $b_1(s)$, $b_2(s)$ indicate the components of the displacement discontinuities across the dislocation line. Substituting eqn (46) into eqns (11) and (12), the temperature potentials are obtained as

$$g_1'(z) = \frac{-i}{2\pi}\int_L b_0(s)\log(z-t)\,\mathrm{d}s + \frac{(k_1-k_2)i}{2\pi(k_1+k_2)}\int_L b_0(s)\log\left(\frac{a^2}{z}-\bar{t}\right)\mathrm{d}s \tag{49}$$

$$g_2'(z) = \frac{-k_1 i}{\pi(k_1+k_2)}\int_L b_0(s)\log(z-t)\,\mathrm{d}s \tag{50}$$

The unknown function $b_0(s)$ in eqns (49) an (50) can be obtained in the sense that the total heat flux across the crack surface must be balanced by the given resultant heat flux Q_1 across the crack surface L in the unflawed media (Chao and Shen, 1995), i.e.

$$Q_1 = -k_1\,\mathrm{Im}[g_1'(t)] + c_0, \quad t \in L \tag{51}$$

where c_0 is a constant to be determined. In addition, the single-valued condition of the temperature must be satisfied, i.e.

$$\int_L b_0(s)\,\mathrm{d}s = 0 \tag{52}$$

Substitution of eqn (49) into eqn (51) yields the singular integral equation together with the subsidiary condition, eqn (52), which may be solved numerically. Having the temperature functions given in eqns (49) and (50), the corresponding stress functions can be also expressed in terms of integral equations by substituting eqns (47)–(50) into eqns (29)–(32). Now, the remaining unknown functions $b_1(s)$ and $b_2(s)$ can be obtained in the sense that the force acting on the crack surface must be balanced by the given resultant force applied on the crack surface (Chao and Shen, 1995), i.e.

$$-Y_1 + iX_1 = \phi_1(t) + t\overline{\phi_1'(t)} + \overline{\psi_1(t)} + c_1 + ic_2, \quad t \in L \tag{53}$$

where c_1 and c_2 are real constants to be determined. Moreover, the requirement of single-valued displacements given by

$$\int_L [b_1(s) + ib_2(s)]\,\mathrm{d}s = \int_L \beta_1\left[\int b_0(\zeta)\,\mathrm{d}\zeta\right]\mathrm{d}s \tag{54}$$

should be satisfied. Thus, the thermoelastic field associated with a single crack located in the matrix can then be obtained once the singular integral eqn (53) together with the subsidiary condition (54) are solved numerically.

For a single crack located in the inclusion, the corresponding homogeneous solutions are given by

$$g_0'(z) = -\frac{i}{2\pi}\int_L b_0(s)\log(z-t)\,\mathrm{d}s \tag{55}$$

$$\phi_0(z) = \frac{iG_2}{\pi(1+\kappa_2)}\int_L [b_1(s) + ib_2(s)]\log(z-t)\,\mathrm{d}s \tag{56}$$

$$\psi_0(z) = \frac{-iG_2}{\pi(1+\kappa_2)}\int_L [b_1(s) - ib_2(s)]\log(z-t)\,\mathrm{d}s - \frac{iG_2}{\pi(1+\kappa_2)}\int_L \frac{[b_1(s)+ib_2(s)]\bar{t}}{z-t}\mathrm{d}s \tag{57}$$

Substituting eqn (55) into eqns (15) and (16), the temperature functions in the matrix and in the inclusion can be respectively expressed as

$$g_1'(z) = \frac{-k_2 i}{\pi(k_1+k_2)} \int_L b_0(s) \log(z-t)\, ds \tag{58}$$

and

$$g_2'(z) = \frac{-i}{2\pi} \int_L b_0(s) \log(z-t)\, ds + \frac{(k_2-k_1)i}{2\pi(k_1+k_2)} \int_L b_0(s) \log\left(\frac{a^2}{z} - \bar{t}\right) ds \tag{59}$$

By a way similar to the previous approach, the unknown constant $b_0(s)$ can be determined numerically. The corresponding stress functions can be also expressed in terms of integral equations by substituting eqns (56)–(59) into eqns (42)–(45). Similarly, the unknown functions $b_1(s)$ and $b_2(s)$ may be solved numerically from eqns (53) and (54) by replacing the stress functions $\phi_1(z)$, $\psi_1(z)$ in eqn (53) with $\phi_2(z)$, $\psi_2(z)$ and the material constant β_1 in eqn (54) with β_2. After having the unknown functions $b_0(s)$, $b_1(s)$ and $b_2(s)$, all the field quantities may be evaluated by means of definite integrals and the resulting stress intensity factors, which are related to the coefficients $b_1(s)$ and $b_2(s)$ (Chen, 1990; Chao and Shen, 1995), will be obtained accordingly.

5. NUMERICAL RESULTS AND DISCUSSIONS

The main purpose of this paper is to study interactions between a crack and a circular elastic inclusion under remote uniform heat flux. This can be achieved by the determination of the stress intensity factors resulted from thermal flux that may properly characterize the interaction behavior between a crack and a circular inclusion. In the following results, only a radial insulated crack (see Fig. 2) located either in the matrix or in the inclusion under remote uniform heat flux, q_0, in the direction approached from the negative y-axis will be considered. The results of nondimensional stress intensity factors vs the distance h/a for different ratios of shear modulus, when a radial crack is embedded in the matrix, are displayed in Fig. 3(a)–(c). In the following discussion, $k_2/k_1 = \beta_2/\beta_1 = 1$, and $\nu_1 = \nu_2 = 0.3$,

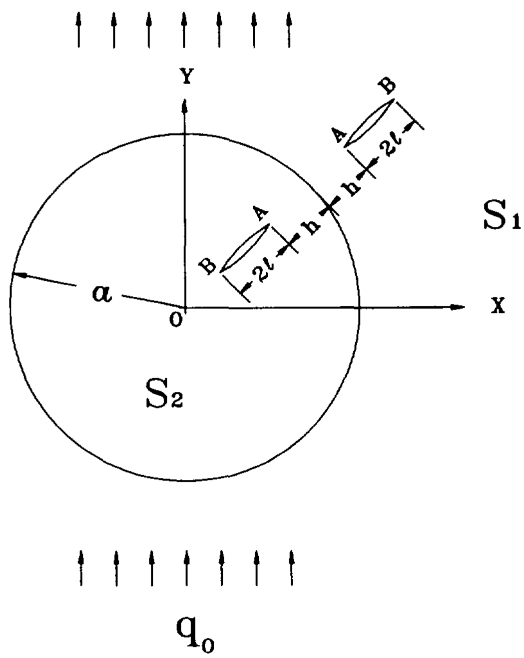
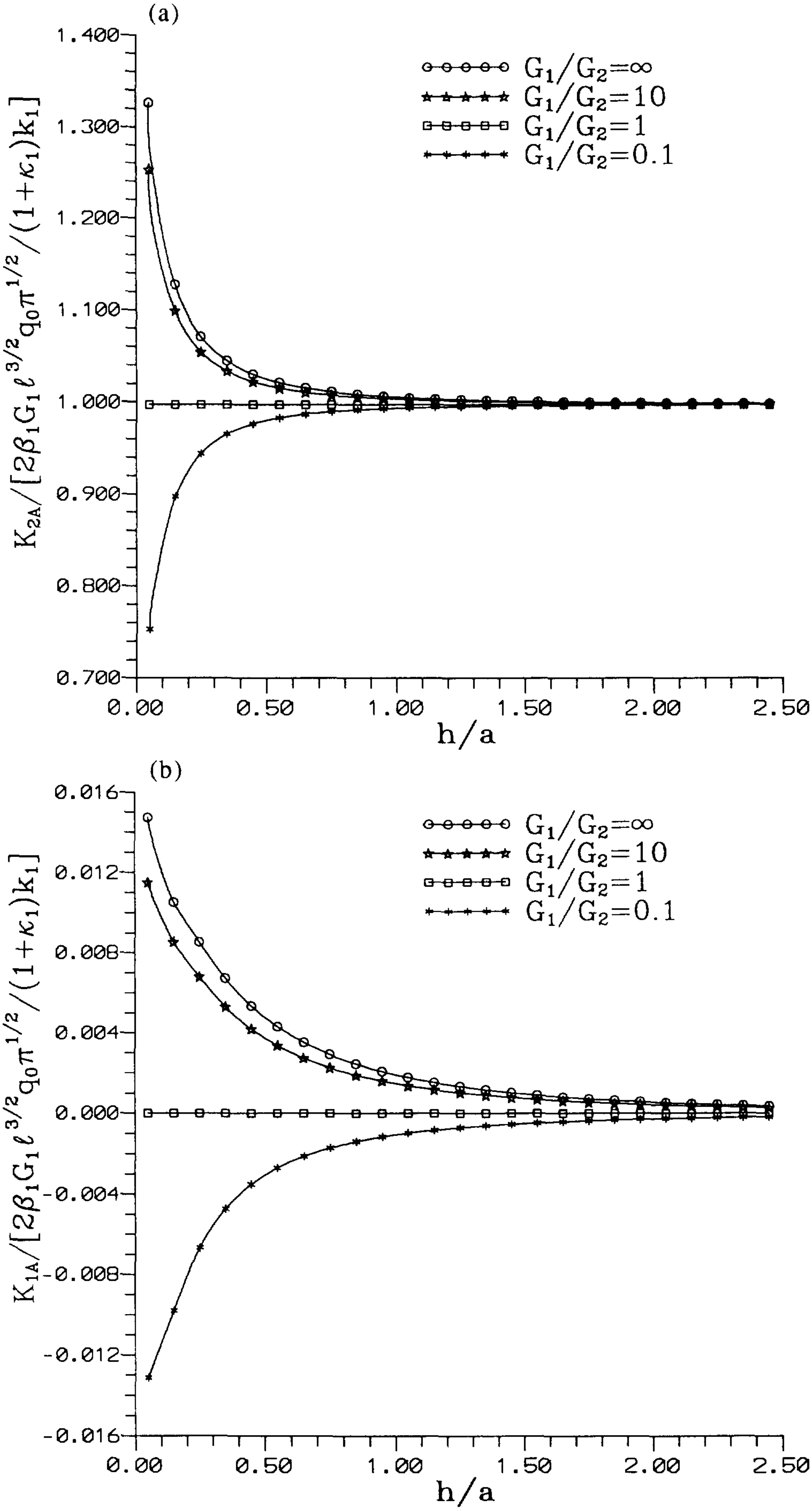


Fig. 2. A radial insulated crack located either in the matrix or in the inclusion.


(a)
$G_1/G_2=\infty$
$G_1/G_2=10$
$G_1/G_2=1$
$G_1/G_2=0.1$
$K_{2A}/[2\beta_1 G_1 \ell^{3/2} q_0 \pi^{1/2}/(1+\kappa_1)k_1]$
1.400
1.300
1.200
1.100
1.000
0.900
0.800
0.700
0.00
0.50
1.00
1.50
2.00
2.50
h/a
(b)
$G_1/G_2=\infty$
$G_1/G_2=10$
$G_1/G_2=1$
$G_1/G_2=0.1$
$K_{1A}/[2\beta_1 G_1 \ell^{3/2} q_0 \pi^{1/2}/(1+\kappa_1)k_1]$
0.016
0.012
0.008
0.004
0.000
-0.004
-0.008
-0.012
-0.016
0.00
0.50
1.00
1.50
2.00
2.50
h/a

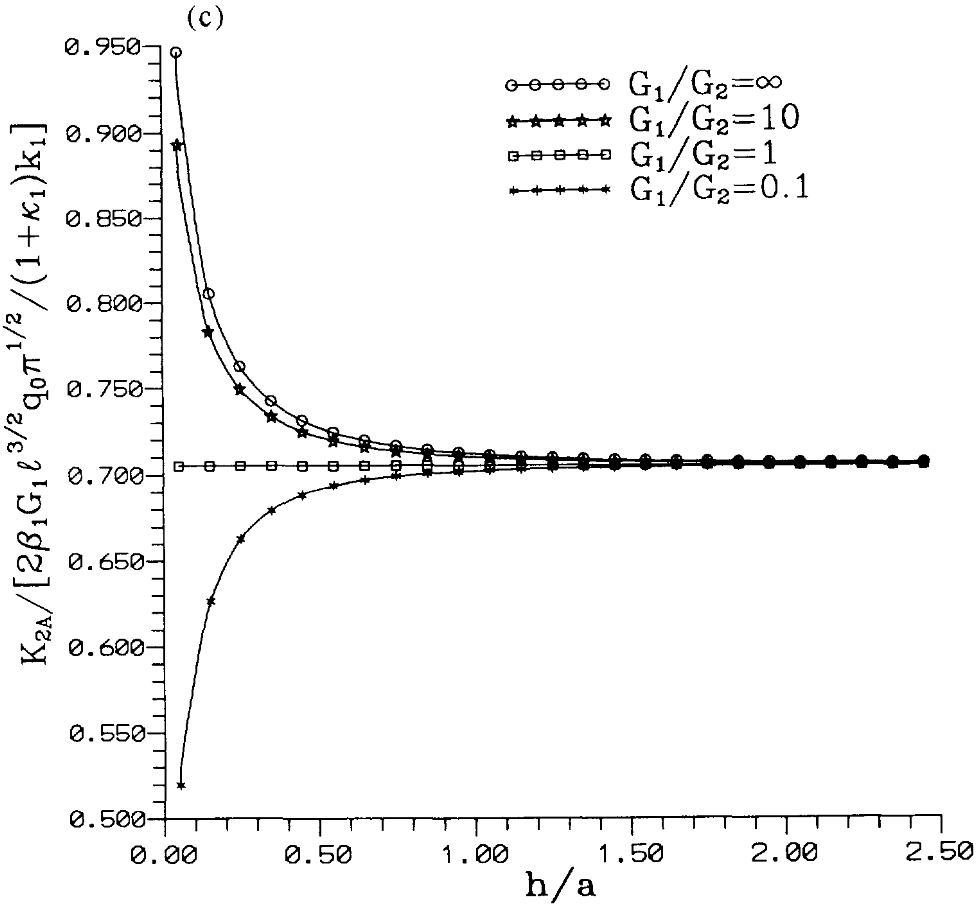
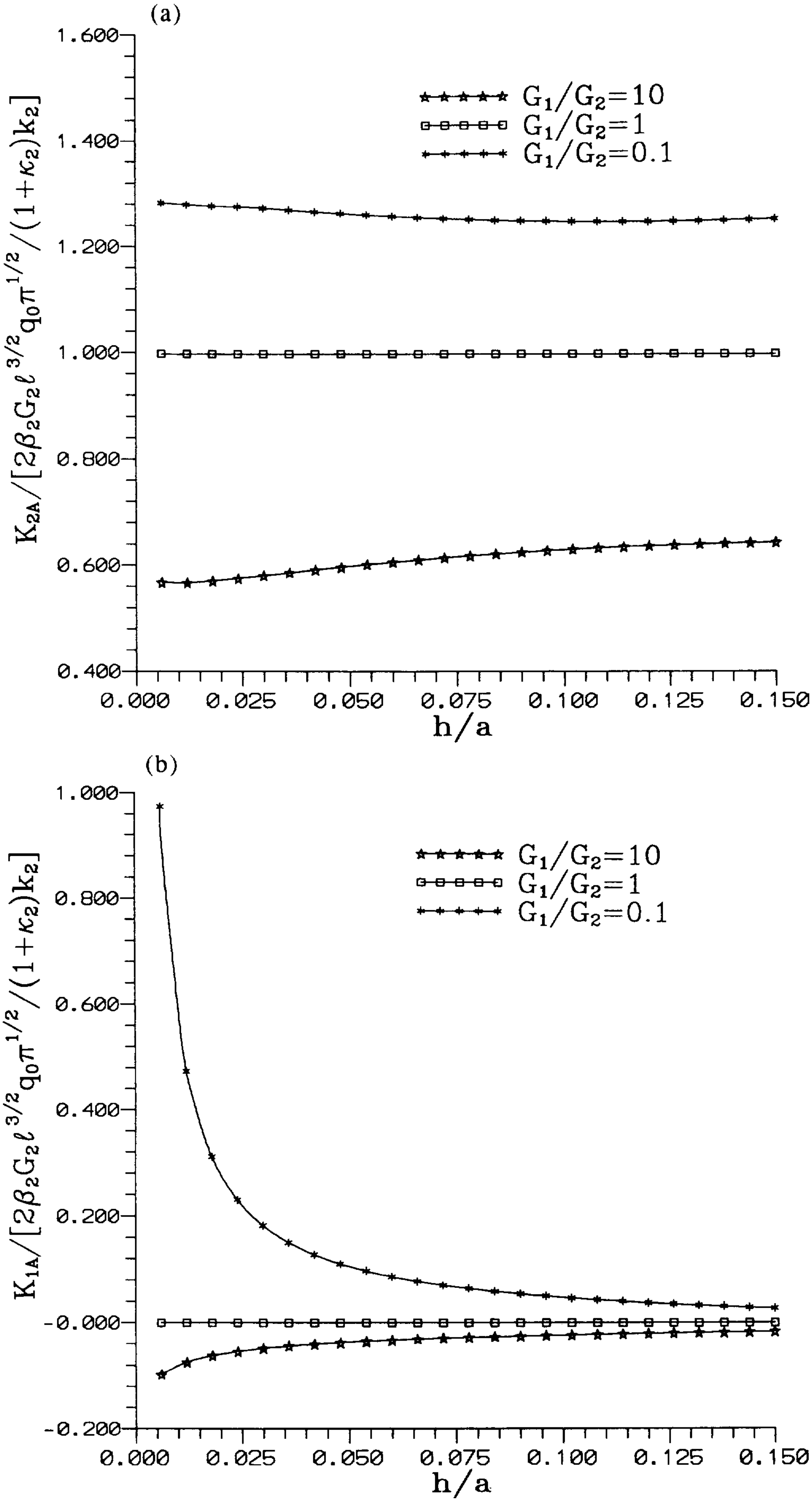


Fig. 3. (a) The mode-II stress intensity factors vs the distance h/a for a radial crack located in the matrix with $\alpha = \beta = 0^{\circ}$. (b) The mode-I stress intensity factors vs the distance h/a for a radial crack located in the matrix with $\alpha = \beta = 45^{\circ}$. (c) The mode-II stress intensity factor vs the distance h/a for a radial crack located in the matrix with $\alpha = \beta = 45^{\circ}$.

$l/a = 0.1$ are used and other values are stated specifically. It is indicated that the positive mode-I stress intensity factors at tip A increase with a decreasing distance h/a as a radial crack is placed in the hard material ($G_1 > G_2$). The trend becomes more obvious for the problems where the inclusion is a hole ($G_1/G_2 = \infty$). On the other hand, the factors K_{1A} will become a negative number as a radial crack is placed in the soft material ($G_1/G_2 = 0.1$). It is understood that stiffer material may tend to act as a barrier in load transfer. It is noted that all the numerical results of stress intensity factors presented in the present study are only for the crack tip closer to the interface because those factors at the crack tip away from the interface are less sensitive to the distance h/a as compared to those at the crack tip close to the interface. It is worth noting that both mode-I and mode-II stress intensity factors vanish as radial crack is placed parallel to the direction of heat flow, i.e. $\alpha = \beta = 90^{\circ}$ because the uniform heat flow would not be disturbed by the presence of an insulated crack. Moreover, the mode-I factors will vanish when a crack is oriented normal to the direction of heat flow, i.e. $\alpha = \beta = 0^{\circ}$ where the heat flux passing along the crack surface will not be disturbed by the presence of a circular inclusion. Similar observations can also be made on the problem when a radial crack is placed in the inclusion as displayed in Fig. 4(a)–(c). Note that all the calculated stress intensity factors are very sensitive to the distance h/a and the convergence of numerical calculation becomes slower when a crack tip approaches the interface. In reality, the use of stress intensity factors can no longer be applied to a very small region near the interface which is beyond a continuum scale level. The results as


(a)
1.600
1.400
1.200
1.000
0.800
0.600
0.400
0.000 0.025 0.050 0.075 0.100 0.125 0.150
h/a
$K_{2A}/[2\beta_2 G_2 \ell^{3/2} q_0 \pi^{1/2}/(1+\kappa_2)k_2]$
$G_1/G_2=10$
$G_1/G_2=1$
$G_1/G_2=0.1$
(b)
1.000
0.800
0.600
0.400
0.200
-0.000
-0.200
0.000 0.025 0.050 0.075 0.100 0.125 0.150
h/a
$K_{1A}/[2\beta_2 G_2 \ell^{3/2} q_0 \pi^{1/2}/(1+\kappa_2)k_2]$
$G_1/G_2=10$
$G_1/G_2=1$
$G_1/G_2=0.1$

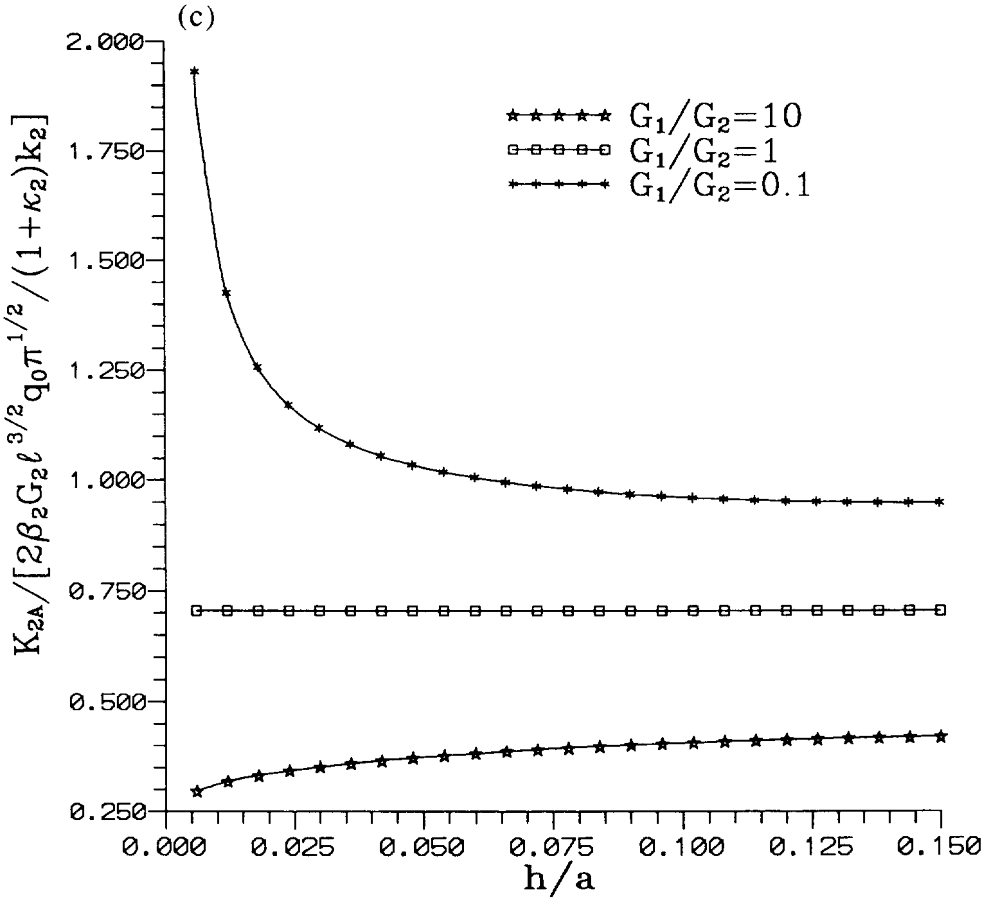


Fig. 4. (a) The mode-II stress intensity factors vs the distance h/a for a radial crack located in the inclusion with $\alpha = \beta = 0^\circ$. (b) The mode-I stress intensity factors vs the distance h/a for a radial crack located in the inclusion with $\alpha = \beta = 45^\circ$. (c) The mode-II stress intensity factors vs the distance h/a for a radial crack located in the inclusion with $\alpha = \beta = 45^\circ$.

indicated in Figs 5–8 are given for three typical examples of composite materials with a radial crack located either in the matrix or in the inclusion. All the thermoelastic properties of composite materials are listed in Table 1.

It is interesting to see that all the mode-I factors for three examples of composite materials are found to be negative as a radial crack embedded in the matrix is oriented at $\alpha = \beta = 45^\circ$ under the given heat flow. This implies that a crack will propagate away from the interface. Conversely, the crack would propagate toward and go through the interface as a radial crack oriented at $\alpha = \beta = 45^\circ$ is placed in the inclusion.

6. CONCLUSIONS

A general solution to the thermoelastic problem of a circular elastic inclusion perfectly bonded to an infinite matrix with a crack located either in the matrix or in the inclusion is presented in this paper. The proposed method is based upon the complex variable theory and dislocation functions which are used to formulate the crack problem. Based upon the analytical continuation theorem, both the temperature and displacement complex potentials are formulated such that the continuity conditions across the interface are satisfied. The resulting singular integral equation with a logarithmic singular kernel (Chen, 1990; Chao and Shen, 1995) are established such that the unknown dislocation functions can then be solved numerically in a straightforward manner.

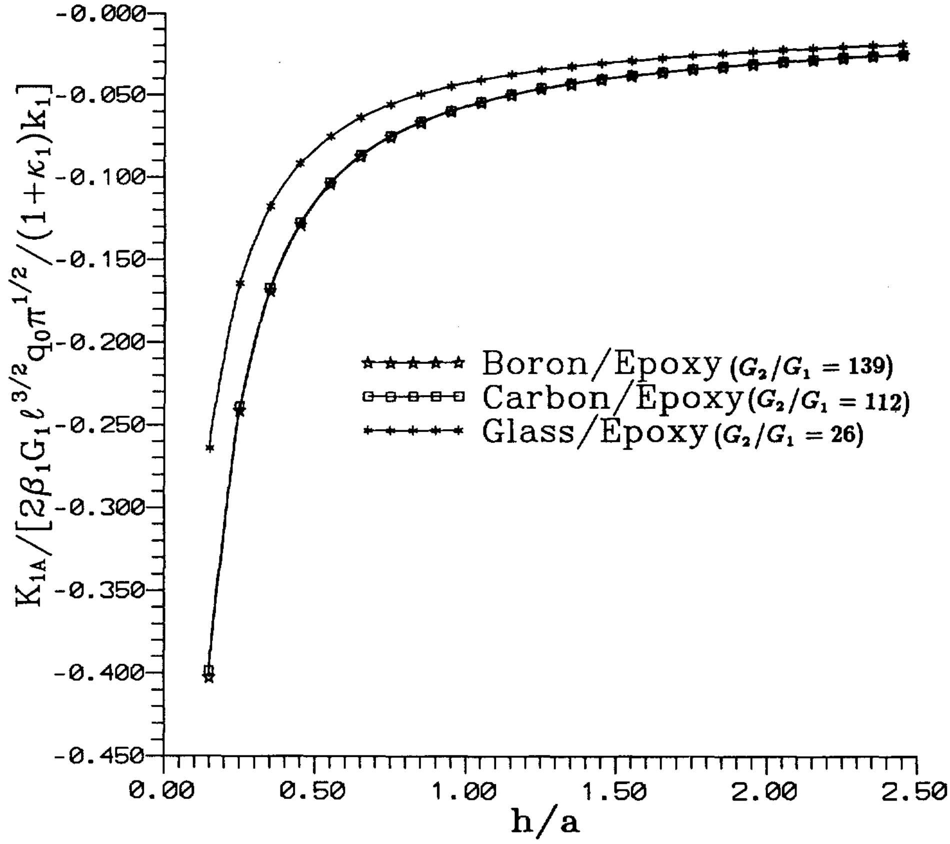


Fig. 5. The mode-I stress intensity factors vs the distance h/a for a radial crack located in the matrix with $\alpha = \beta = 45°$.

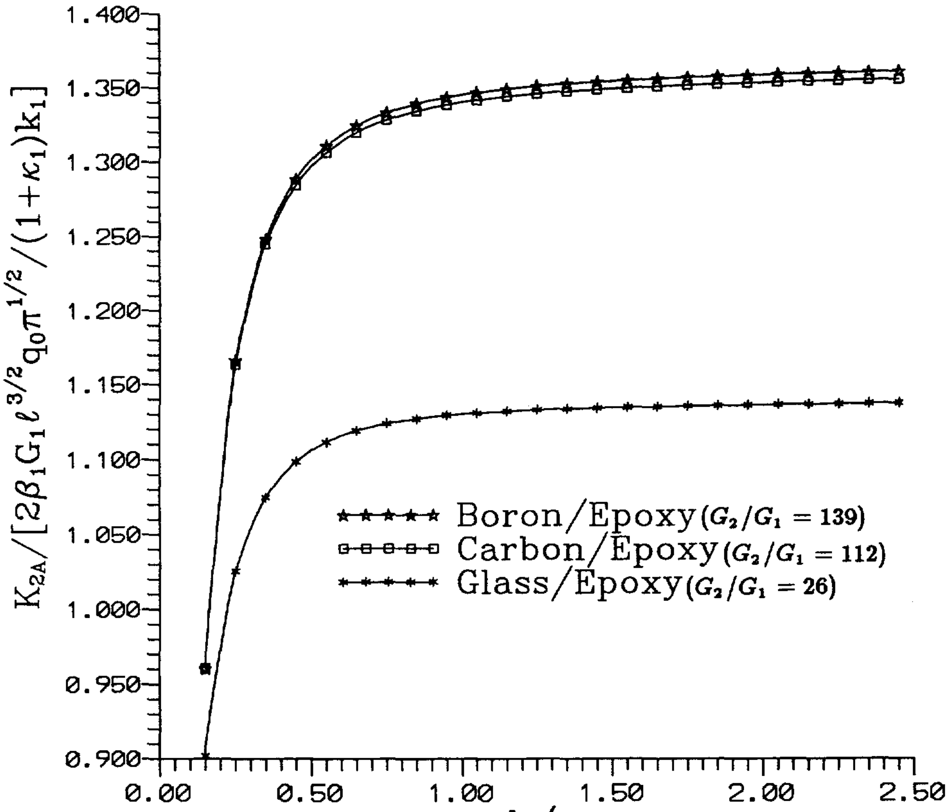


Fig. 6. The mode-II stress intensity factors vs the distance h/a for a radial crack located in the matrix with $\alpha = \beta = 45°$.

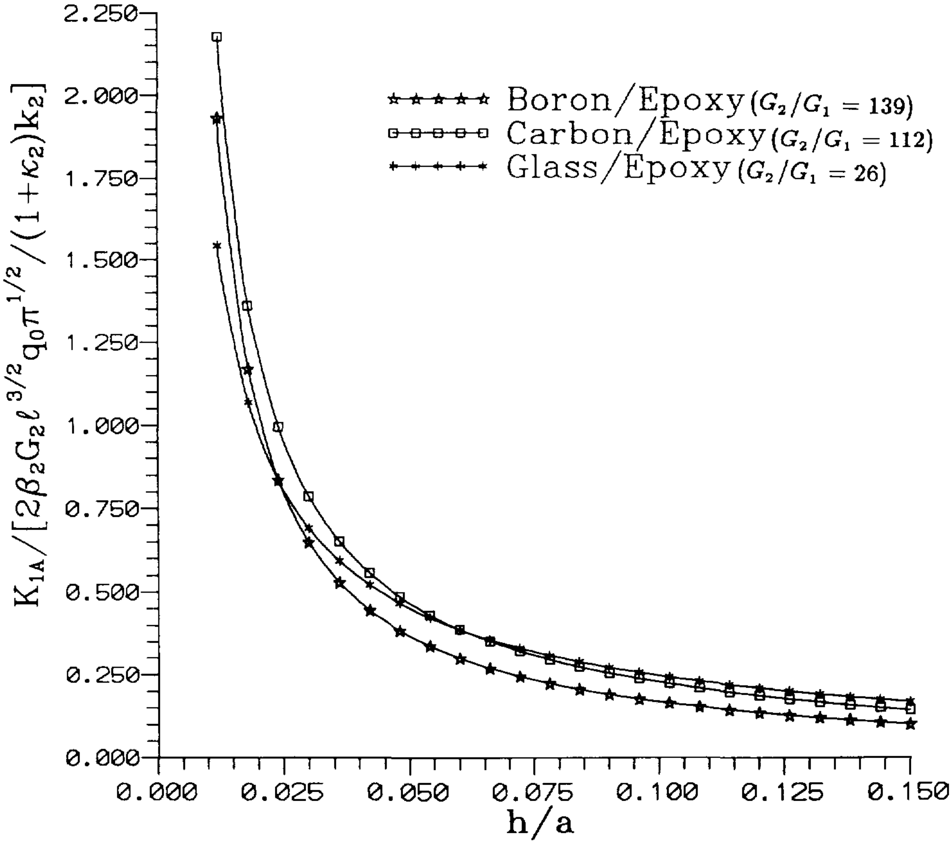


Fig. 7. The mode-I stress intensity factors vs the distance h/a for a radial crack located in the inclusion with $\alpha = \beta = 45°$.

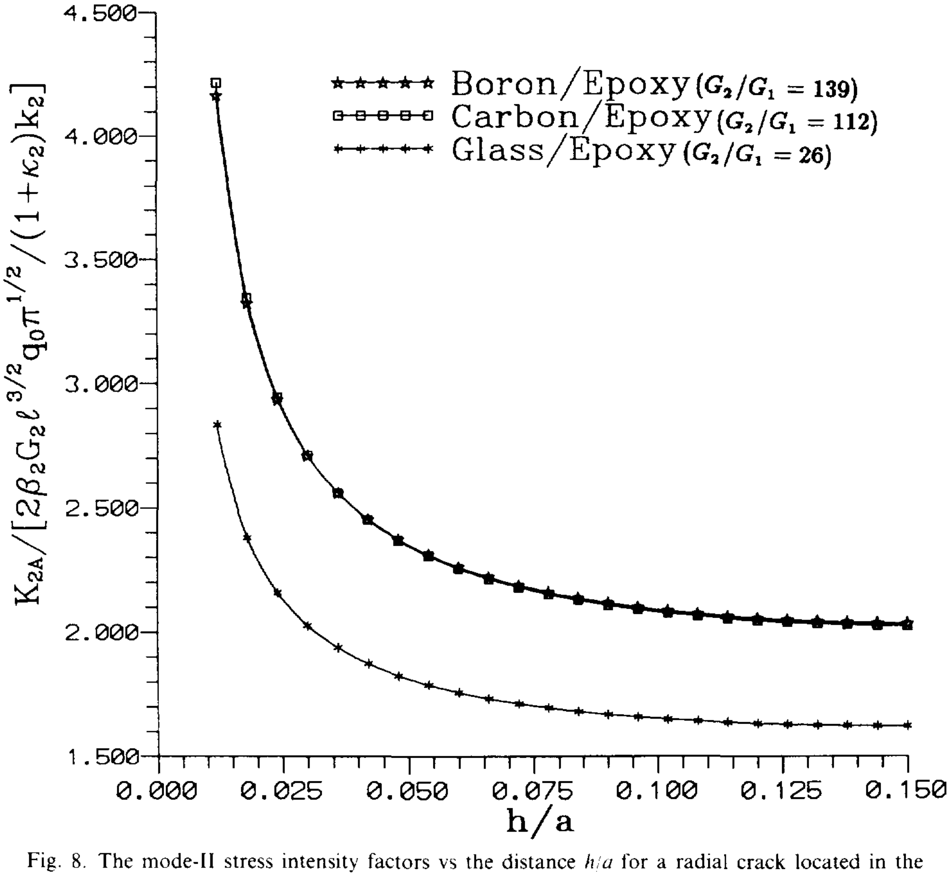


Fig. 8. The mode-II stress intensity factors vs the distance h/a for a radial crack located in the inclusion with $\alpha = \beta = 45°$.

Table 1. Typical properties of materials for composites

Properties	Inclusion			Matrix
	Boron	Carbon	Glass	Epoxy
Shear modulus, GNm^{-2}	172.5	138.7	31.74	1.24
Poisson's ratio	0.2	0.2	0.2	0.4
Thermal expansion coefficient, 10^{-6} $^{\circ}C^{-1}$	5.0	2.7	5.0	57.6
Heat conductivity, $W (m\ ^{\circ}C)^{-1}$	18.2	15.6	1.94	0.45

For studying the thermoelastic interactions between a crack and a circular elastic inclusion, the stress intensity factors for several numerical examples are computed which are found to depend upon material properties of matrix or inclusion, crack orientation and geometric configuration. It is emphasized that the method presented in this work can also be applied to more complicated problems with irregularly shaped multiple cracks embedded either in the matrix or in the inclusion.

REFERENCES

Atkinson, C. (1972). The interaction between a crack and an inclusion. *Int. J. Engng Sci.* **10,** 127–136.

Chao, C. K. and Shen, M. H. (1995). Solutions of thermoelastic crack problems in bonded dissimilar media or half-plane medium. *Int. J. Solids Structures* **32,** 3537–3554.

Chen, Y. Z. (1990). New integral equation for curve crack problem in plane elasticity with arbitrary loading condition. *Int. J. Fracture* **46,** R43–R46.

Dundurs, J. and Mura, T. (1964). Interaction between an edge dislocation and a circular inclusion. *J. Mechanics and Physics of Solids* **12,** 177–189.

Dundurs, J. and Sendeckyj, G. P. (1965). Edge dislocation inside a circular inclusion. *J. Mechanics and Physics of Solids* **13,** 141–147.

Erdogan, F. and Gupta, G. D. (1975). The inclusion problem with a crack crossing the boundary. *Int. J. Fracture* **11,** 13–27.

Honein, T. and Herrmann, G. (1988). The involution correspondence in plane elastostatics for regions bonded by a circle. *J. Appl. Mech.* **55,** 566–573.

Honein, T. and Herrmann, G. (1990). On bonded inclusions with circular or straight boundaries in plane elastostatics. *J. Appl. Mech.* **57,** 850–856.

Hutchinson, J. W. (1987). Crack tip shielding by micro-cracking in brittle solids. *Acta Metallurgica* **35,** 1605–1619.

Li, R. and Chudnovsky, A. (1994). The stress intensity factor Green's function for a crack interacting with a circular inclusion. *Int. J. Fracture* **67,** 169–177.

Patton, E. M. and Santare, M. H. (1993). Crack path prediction near an elliptical inclusion. *Engng Fracture Mech.* **44,** 195–205.

Sekine, H. (1977). Crack problem for a semi-infinite solid with heated bounding surface. *J. Appl. Mech.* **44,** 637–642.